ON A PARTICULAR FOURIER INVERSION

by

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A function f(x) is the Fourier transform of g(x) if

$$f(x) = (1/2\pi)^{0.5} \int_{-\infty}^{\infty} g(t)e^{itx}dt \qquad \ldots 1)$$

Under suitable conditions on g(x), it then follows that

$$g(x) \simeq (1/2\pi)^{0.5} \int_{-\infty}^{\infty} f(t)e^{-itx}\, dt \qquad \ldots 2)$$

where the value of the right member is

$$\lim_{h \to 0} \tfrac{1}{2}[g(x + h) + g(x - h)] \qquad \ldots 3)$$

if g(x) is of bounded variation in the neighborhood of x. Such functions f(x) and g(x) are sometimes said to be a pair of Fourier transforms.

Inversion formulae are solutions of the integral equation 2). Conditions must be imposed on f(t) and on the path of integration of the contour integrals.

In formulating a solution to a particular problem, it becomes necessary to evaluate the Fourier inversion expression

$$f(x) = (1/2\pi)^{0.5} \int_{-\infty}^{\infty} F(u)e^{-iux}\, du \qquad \ldots 4)$$

which is identical to equation 2) save for the dummy variable of integration, where the integrand F(u) is given by

$$F(u) = u\, G(u)\ln(u + a). \qquad \ldots 5)$$

We rewrite equation 5) as

$$F(u) = u^{m}(u + a)\bar{G}(u)\bar{H}(u) \qquad \ldots 6)$$

where

$$\bar{H}(u) = [1/(u + a)]\ln(u + a) \qquad \ldots 7)$$

and GH is the transform of a product. According to Bateman, Ref. 1, the inversion of this part is

$$[i^{m}/(2\pi)^{0.5}]\, \frac{\partial^{m}}{\partial x^{m}} \left(i\frac{\partial}{\partial x} + a\right) \int_{-\infty}^{\infty} G(\xi)H(x - \xi)d\xi \qquad \ldots 8)$$

The function H(x) is obtained from Titchmarch, Ref. 2, as

$$H(x) = [1/(2\pi)^{0.5}] \int_{-\infty + iu''}^{\infty + iu''} \bar{H}(u)e^{iux}\, dx \qquad \ldots 9)$$

Let

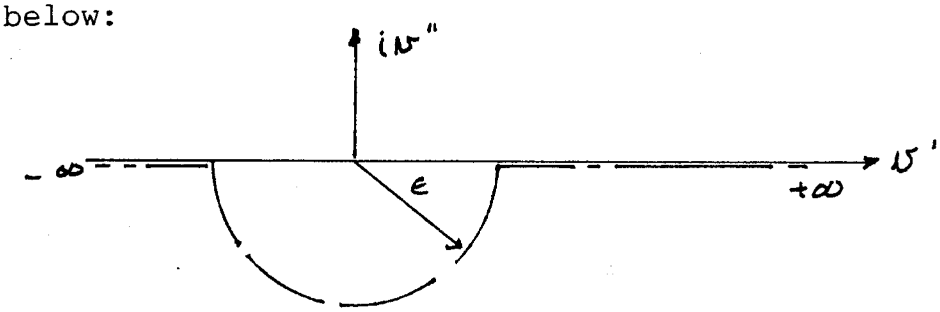
$$u + a = v \qquad \ldots 10)$$

$$v = v' + iv'' \qquad \ldots 11)$$

Then

$$H(x) = [1/(2\pi)^{0.5}]e^{iax} \int_{-\infty + iv''}^{\infty + iv''} (1/v)\ln v\; e^{ivx}\, dv \qquad \ldots 12)$$

Since the only pole of the integral is situated at v = 0, it is possible to choose the path of integration according to the figure below:



and then let $\epsilon \rightarrow 0$. For logv, the branch is chosen which is purely real along the positive abscissa to make logv single valued. For H(x) this results with $v = \epsilon e^{i\theta}$ along the semi-circle.

References

1. Bateman, Harry, Higher Transcendental Functions, McGraw Hill Book Co., NY, 1953
2. Titchmarsh, E.C., Theory of Functions, Oxford University Press, 1932